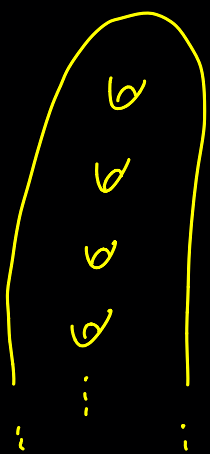
Most Big Mapping Class Gps fail the Tits Alternative

Daniel Allcock, U.T. Geometry Seminar Mar 4, 2021.

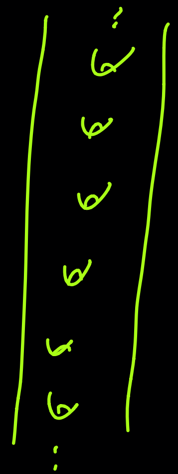
Big Surface: π_1 not finitely generated. (All surfaces taken to be connected. FOR EXPOSITORY PURPOSES (ONLY): orb'l & w/o ∂.)

"Loch Ness Monster"



End space: 1 end, approximated by genus ✓

"Jacob's ladder"



End space: 2 ends, each approx. by genus.

"Cantor Octopus Surface"

S^2 - Cantor set

End space: ∞ many ends: end space is a Cantor Set. None of them is approximated by genus.

Classification of surfaces without ∂

Requires concept of an end being approximated by genus if every neighborhood of that end has ∞ genus.

$\boxed{\text{Thm (Kerékjártó 1923)}}$ given surfs S_1, S_2 of same genus. Then

$S_1 \cong S_2 \iff \exists$ homeo $\mathrm{Ends}(S_1) \cong \mathrm{Ends}(S_2)$ that identifies the ends approximated by genus of S_1 with those of S_2.

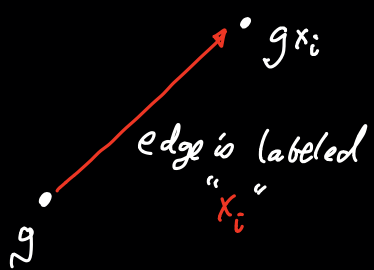
<u>Mapping class gp of S</u>: def'd the usual way: $\mathrm{Homeo}^+(S)/\simeq$

For cpt S, $MCG(S)$ is an industry. For big S, MCG can be <u>very</u> big:
Aougab-Patel-Vlamis found uncountably many S with property:

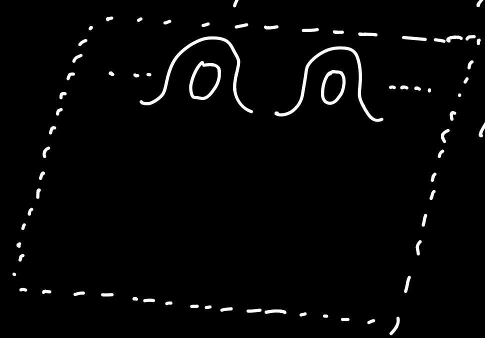
Every countable gp occurs as a subgp of $MCG(S)$!

<u>Sketch for S = Loch Ness Surface</u>: given countably ∞ G, choose generating set $X = G$; build Cayley graph of G:

$\forall g \in G$, $\forall x_i \in X$

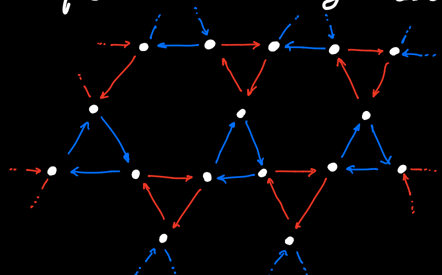


$V :=$ "vertex surface" $= \mathbb{R}^2$ with ∞ many handles, minus ∞ many open disks:



minus

⌜example with finite gen set:



$1 = a^3 = b^3 = (ab)^3$ ⌟

One copy of V per vertex, ∂ circles glued as in Cayley graph. $S :=$ result.

$G \curvearrowright S$ ✓

S has 1 end, approximated by genus.

Kerékjártó $\Longrightarrow S \cong$ Loch Ness surf.

⌜RK: "hyperbolic surfaces with prescribed ∞ symmetry gps" (A., 2006): can choose hyperbolic metric st $\mathrm{Isom}(S)$ <u>equals</u> G. ⌟

So: expect big MCGs to be "wild". $\exists$ many measures of wild:

$\boxed{\text{Thm (Tits)}}$ Every finitely gen'd subgp G of GL_n(any field k) either (i) is solvable (up to finite index) ["G small"]

or (ii) contains a copy of the free gp F_2. ["G big"]

Express by saying "$GL_n(k)$ satisfies the Tits Alternative"

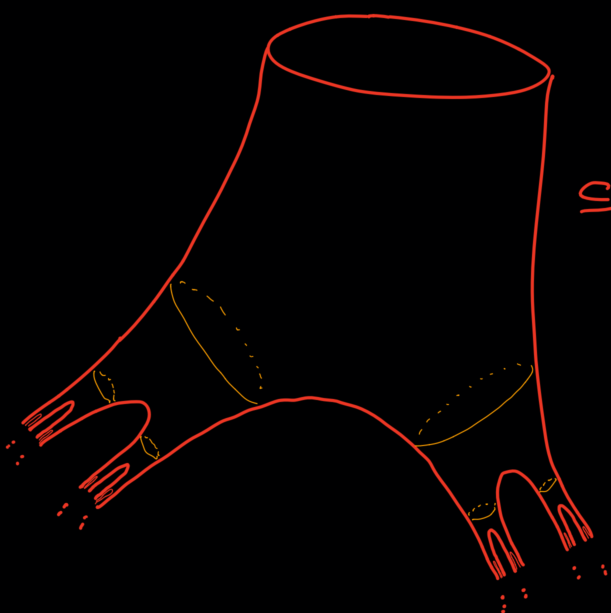
Impt. Gps Satisfying it: MCG(cpt surf)
$Out(F_n)$
Gromov-Hyperbolic gps.
Various Artin Gps (incl. Braid Gps)

Lanier-Loving showed a big MCG can fail the "strong" Tits Alt.
Asked about the ordinary one:

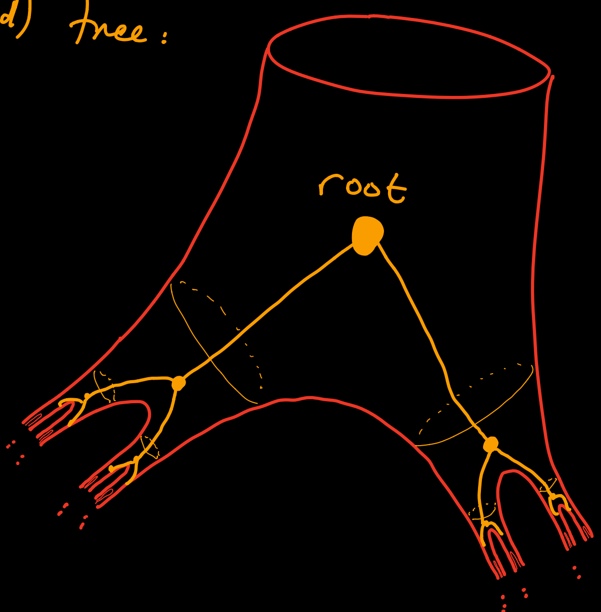
$\boxed{\text{Thm. (A. 2020)}}$ Suppose S is a big surface (connected, orientable, without ∂). Then $MCG(S)$ fails the Tits Alternative.

Essential Case: S contains $D^2 - ($Cantor set in interior of $D^2)$:

Cantor Octopus (minus open disk) decomposed into pairs of pants, glued in the pattern of the ∞ binary (rooted) tree:



$\subseteq S$



We must build a f.g. subgp $\tilde{G} \leq MCG(S)$ that is (i) has no solvable finite-index subgp, (ii) contains no copy of F_2.

$\exists$ a famous f.g. gp G ("Grigorchuk's gp") satisfying (i) & (ii).

Our $\tilde{G}$ fits into an exact sequence

$$1 \longrightarrow (\text{abelian}) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow 1$$

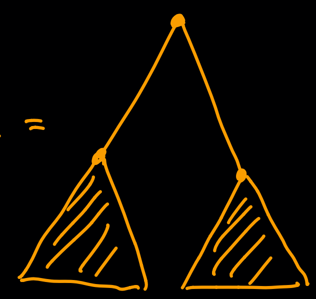
∴ $\hat{G}$ also satisfies (i), (ii), so MCG(S) fails Tits Alternative.

$\boxed{\text{Grigorchuk's gp } G}$ f.g., ∞, torsion, def'd $\leq \mathrm{Aut}(T :=$ rooted bin. tree$)$

($\exists$ other famous properties.) Generators:

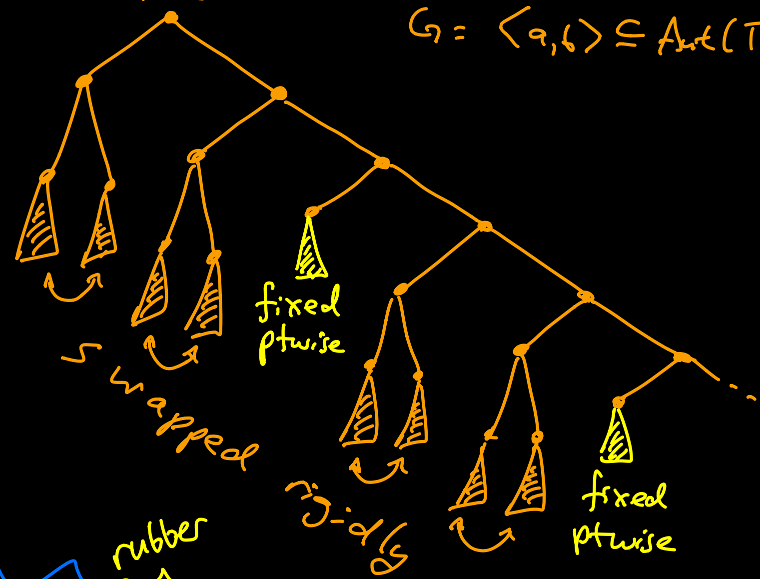
$G = \langle a, b \rangle \leq \mathrm{Aut}(T)$

$a =$

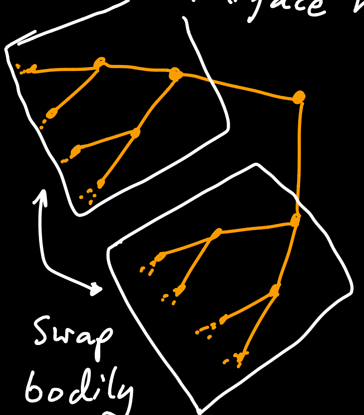


Swap the subtrees rigidly

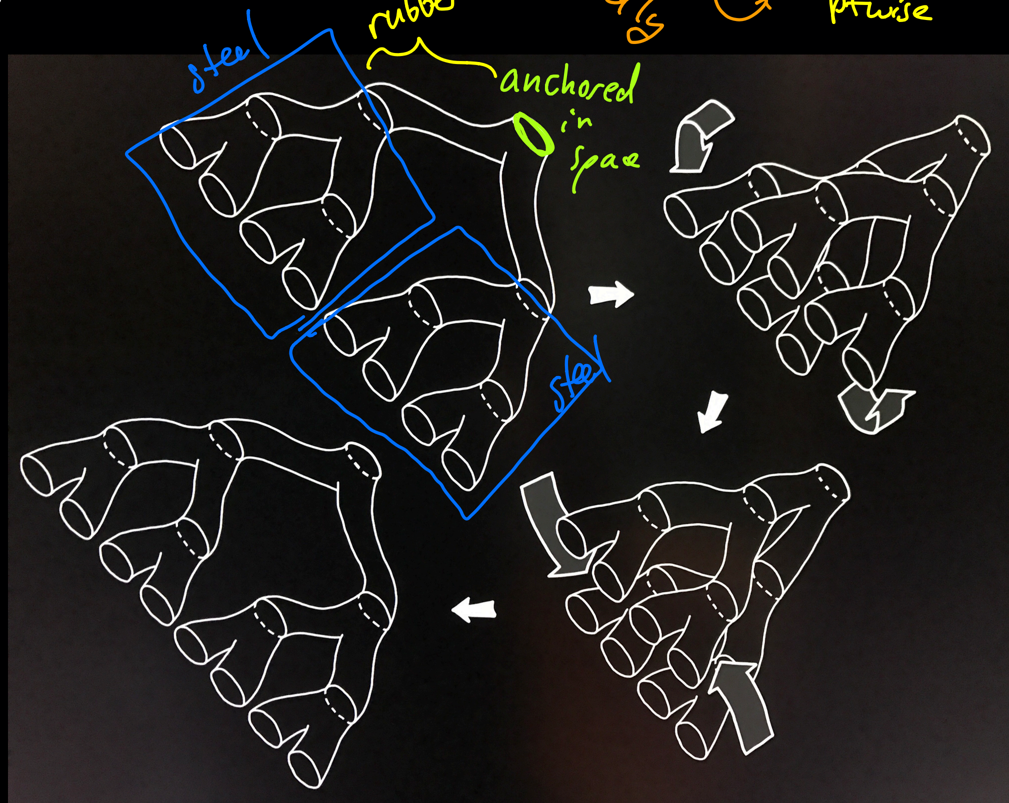
$b =$



The key idea: interpret the swap of 2 subtrees as a surface homeo.



" = "



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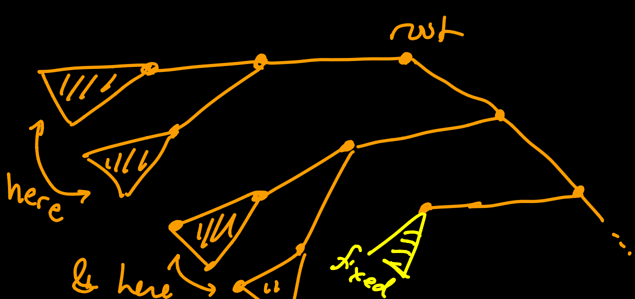
Take the end result of this isotopy in $\mathbb{R}^3$ as defn of $\hat{a} \in \mathrm{Homeo}^+(S)$.

Similarly for $\hat{b}$, at each vertex whose children get swapped. $\hat{G} := \langle \hat{a}, \hat{b} \rangle \leq \mathrm{Homeo}^+(S)$.



$\tilde{a}, \tilde{b} :=$ image of $\hat{a}, \hat{b}$ in MCG(S)

$\tilde{G} := \langle \tilde{a}, \tilde{b} \rangle =$ image of $\hat{G}$.

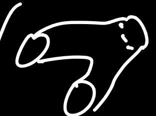
Pf of $1 \to (\text{abelian}) \to \tilde{G} \to (\text{Grigorchuk's } G) \to 1$

Recall
$$\begin{array}{ccccc} \hat{G} & \longrightarrow & \tilde{G} & \xrightarrow[\{\text{ends of } D^2 - (\text{Cantor})\}]{\text{define } q := \text{action on}} & G \\ \cap\!\! | & & \cap\!\! | & & \\ \mathrm{Homeo}^+(S) & & MCG(S) & & \end{array}$$

$\{\text{ends of } D^2-(\text{Cantor})\} = \{\text{ends of } T\}$

$\hat{q} :=$ this composition.

Why is $\ker(q) \leq \tilde{G}$ abelian? Really we study $\ker(\hat{q})$. So say $\hat{g} \in \hat{G}$ acts triv. on $\{\text{ends}\}$. Permutes pairs of pants (since $\hat{a}, \hat{b}$ do). $\therefore$ Sends each to itself; indeed each waist & cuff maps to self.

Use: $MCG($  $)$ is gen'd by the Dehn twists around ∂ loops.

[Note: MCG of bdd surf def'd to fix ∂ ptwise.]

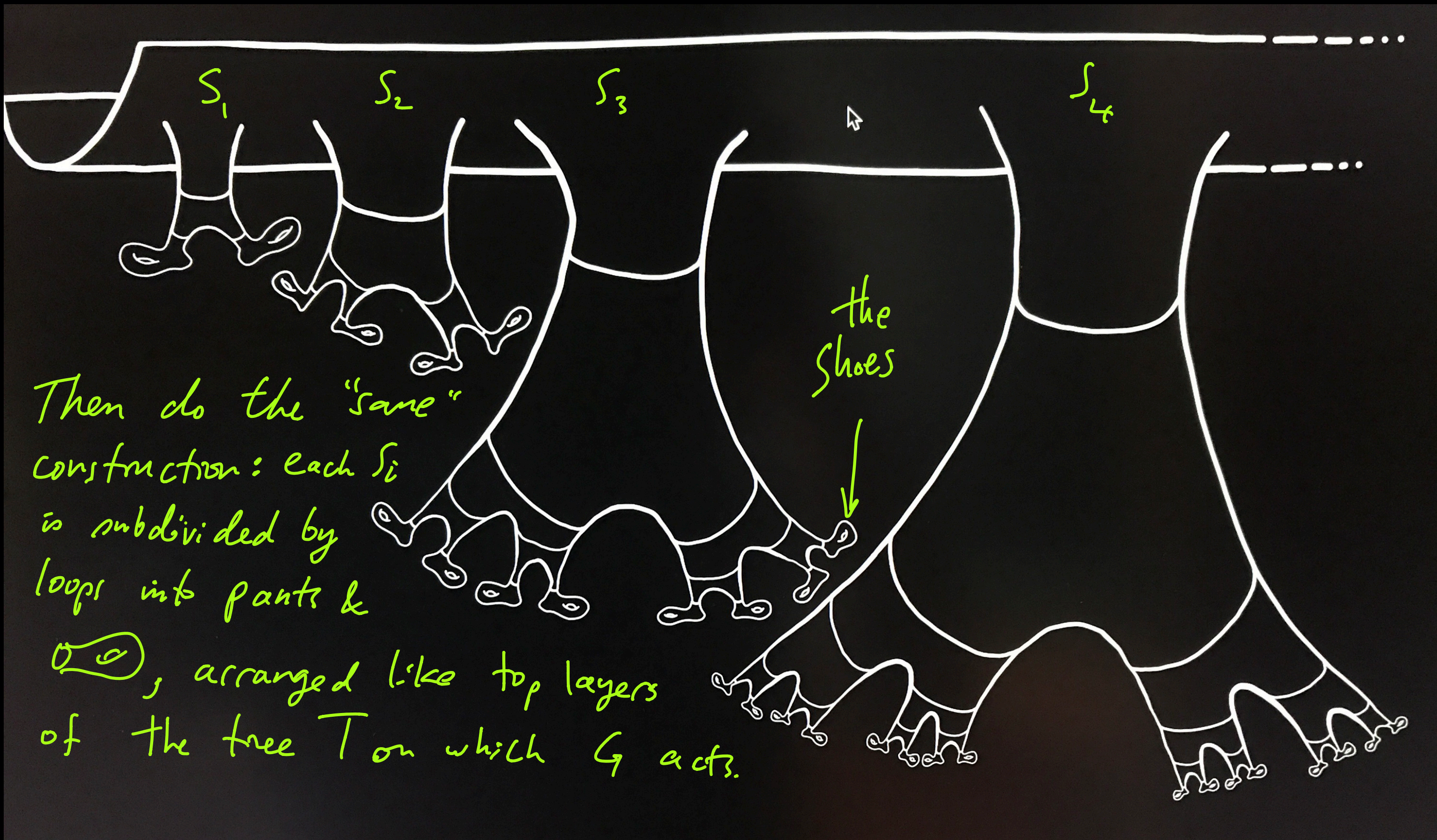
$\therefore$ $\hat{g} \simeq$ product of twists around some waists & cuffs (maybe ∞ly many).

These all commute in $\tilde{G} \leq MCG(S)$, so $q: \tilde{G} \to G$ has abel. kernel. $\boxed{\text{QED}}$

For case of gen'l S, use:

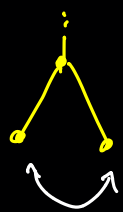
$\boxed{\text{Lemma}}$ Any (connected, or'b'l, w/o ∂) big surface has one of the features:
(i) ∞ genus
(ii) ∞ly many punctures | isolated ends: 
(iii) $D^2 - (\text{Cantor set})$. | ∞ many punctures + 1 non-isolated end

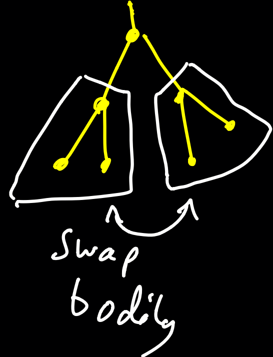
So it remains to prove cases (i), (ii). Arguments are essentially the same. (i) Suppose S has ∞ genus. Decompose S like this:

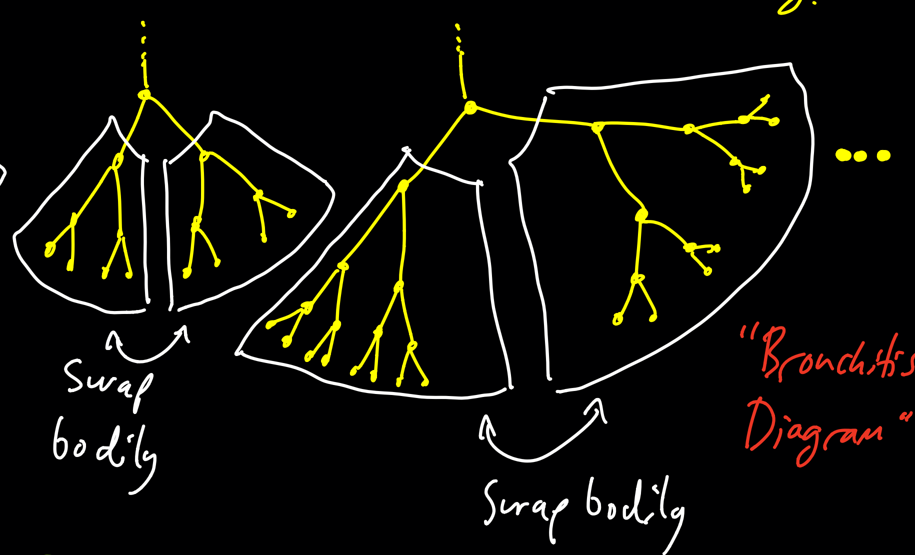


Then do the "same" construction: each S_i is subdivided by loops into pants & , arranged like top layers of the tree T on which G acts.

E.g., $\hat{a}$ acts on S, so as to permute the pants & shoes by:







& $\hat{b}$ similarly

As before, $\hat{G} = \langle \hat{a}, \hat{b} \rangle$. $\tilde{a}, \tilde{b}$ = images in $MCG(S)$; $\tilde{G} = \langle \tilde{a}, \tilde{b} \rangle$. Essentially same argument as before $\Longrightarrow$ $\tilde{G} \longrightarrow \mathrm{Aut}(H_1(S))$ has abelian kernel & image = Grigorchuk's G. $\boxed{QED}$

Thank you for your attention!